probable that this excess of negative velocity on the part of the lines of these elements of high atomic weight is to be ascribed to relative upward motion of the gases at the low levels at which they occur in the stellar atmosphere. These results accordingly may be interpreted as affording further evidence for the presence of convection currents such as have been inferred from solar and stellar observations by St. John and Babcock[5] and St. John and Adams.[6]

[1] Adams and Joy, *Pub. Astron. Soc. Pacific*, October, **1926.**

[2] Cecilia H. Payne, *Harvard College Observatory Bulletin*, 841.

[3] S. A. Mitchell, *Astrophys. J.*, **38**, 407, 1913.

[4] *Mt. Wilson Contr.*, No. 50; *Astrophys. J.*, **33**, 64, 1911.

[5] *Mt. Wilson Contr.*, No. 278; *Astrophys. J.*, **60**, 32, 1924.

[6] *Mt. Wilson Contr.*, No. 279; *Astrophys. J.*, **60**, 43, 1924.

GROUPS OF COLLINEATIONS IN A SPACE OF PATHS

By M. S. Knebelman

Princeton University

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1. In a paper by Professor L. P. Eisenhart and the present writer entitled[1] "Displacements in a Geometry of Paths" we have obtained the equations that the components of an infinitesimal transformation must satisfy in order to carry the paths of an affine manifold into paths. We called such transformations *infinitesimal displacements* because from one point of view they are generalizations of *motion* in Riemannian geometry. However, Professor Eisenhart has pointed out that these transformations should be called *collineations*, since he has shown that for a projectively flat space, referred to a suitable coördinate system, these transformations are collineations. We shall, therefore, adopt this name for any space of paths. If the transformation is such that the affine parameter of the paths is preserved, it will be called an *affine collineation*, otherwise a *projective collineation*.

In the present note we shall obtain the conditions under which an affine manifold admits an r-parameter finite continuous group of collineations, stating the results and indicating the methods in the barest outline, leaving the formal proofs to a subsequent publication.

2. Since all projectively related manifolds admit the same projective collineations we choose one[2] for which the generalized Ricci tensor R_{ij} is symmetric and, therefore, the alternating contracted curvature tensor $S_{ij} = 0$. The equations of condition then become

$$\xi^i_{,j,k} - \xi^l B^i_{jkl} = \delta^i_j \varphi_{,k} + \delta^i_k \varphi_{,j}, \tag{2.1}$$

where

$$\varphi_{,j} = \frac{1}{n+1} \xi^h_{,h,j}.$$

The conditions of integrability of (2.1) are:

$$\xi^h B^i_{jkl,h} - \xi^i_{,h} B^h_{jkl} + \xi^h_{,j} B^i_{hkl} + \xi^h_{,k} B^i_{jhl} + \xi^h_{,l} B^i_{jkh} = \delta^i_l \varphi_{,j,k} - \delta^i_k \varphi_{,j,l}. \tag{2.2}$$

When these are contracted for i and l, we find

$$\varphi_{,j,k} = \frac{1}{n-1} (\xi^h R_{jk,h} + \xi^h_{,j} R_{kh} + \xi^h_{,k} R_{jh}), \tag{2.3}$$

so that (2.2) takes the form

$$\xi^h W^i_{jkl,h} - \xi^i_{,h} W^h_{jkl} + \xi^h_{,j} W^i_{hkl} + \xi^h_{,k} W^i_{jhl} + \xi^h_{,l} W^i_{jkh} = 0, \tag{2.4}$$

where W^i_{jkl} is the Weyl projective curvature tensor. The necessary and sufficient condition that (2.1) should form a completely integrable system, is that (2.4) be satisfied identically; this is equivalent to $W^i_{jkl} = 0$. In case $n \neq 2$, the vanishing of the Weyl tensor also makes (2.3) completely integrable. If $n = 2$, $W^i_{jkl} \equiv 0$ but the complete integrability of (2.3) depends upon the vanishing of the projective covariant.[3] Therefore, for all values of n (2.1) will be completely integrable only if the space is projectively flat.

If $W^i_{jkl} \neq 0$, we build sets of equations for the algebraic determination of the unknown quantities ξ^i, $\xi^i_{,j}$ and $\varphi_{,j}$. Calling these sets $F^{(1)}$, $F^{(2)} \ldots$, we may write them in the form

$$F^{(\alpha)}; \quad \xi^i D^{(\alpha)}_i + \xi^i_{,j} C^{(\alpha)j}_i + \varphi_{,j} E^{(\alpha)j} = 0 \quad (\alpha = 1, 2, \ldots), \tag{2.5}$$

where $D^{(\alpha)}$, $C^{(\alpha)}$ and $E^{(\alpha)}$ are tensors depending upon B^i_{jkl}, W^i_{jkl} and their covariant derivatives of orders not higher than α.

Professor O. Veblen and Dr. J. M. Thomas[4] have established a general existence theorem for systems of partial differential equations of the first order. For the particular problem under discussion this may be expressed in the form:

The necessary and sufficient condition that (2.1) admits a solution, is that a positive integer N (≥ 1) exist, such that the rank of the matrices of $F^{(1)} \ldots$, $F^{(N)}$ and of $F^{(1)} \ldots, F^{(N+1)}$ be the same.

If the condition of the theorem is satisfied and the rank of the matrices is $n^2 + 2n - r$, the general solution of (2.1) will depend on r arbitrary parameters. Assuming this to be the case and letting $\xi^i_{(\alpha)}$ ($\alpha = 1, \ldots, r$) be the r independent solutions of (2.1), any other must be of the form $a^\alpha \xi^i_{(\alpha)}$, where the a's are arbitrary constants.

If $\xi^i_{(\alpha\beta)}$ are the components corresponding to the alternant of two generators, (X_α, X_β), they satisfy (2.1) in virtue of (2.4) and of the fact that

$$\xi^h_{(\alpha\beta),h,j} = (\xi^k_{(\alpha)}\varphi_{(\beta),k})_{,j} - (\xi^k_{(\beta)}\varphi_{(\alpha),k})_{,j}.$$

This fact, coupled with the first theorem in our previous paper, gives the theorem:

If (2.1) admits r independent solutions, the manifold admits an r-parameter finite continuous group of projective collineations.

The greatest number of parameters possible in a group of projective collineations is $n^2 + 2n$ and only projectively flat spaces admit such a group. In this case, if the coördinates are cartesian, the coefficients of projective connection are zero and (2.1) reduce to

$$\frac{\partial^2\xi^h}{\partial x^i\partial x^j} = \frac{1}{n+1}\left(\delta^h_i \frac{\partial^2\xi^k}{\partial x^k\partial x^j} + \delta^h_j \frac{\partial^2\xi^k}{\partial x^k\partial x^i}\right), \tag{2.6}$$

the solution of which gives

$$\xi^i = x^i a_j x^j + b^i_j x^j + c^i \tag{2.7}$$

the finite transformations generated being of the form

$$x'^i = \frac{A^i_j x^j + B^i}{1 + C_k x^k}$$

which is the result to which we referred in §1.

3. The problem of affine collineations is solved by considering the equations

$$\xi^i_{,j,k} = \xi^l B^i_{jkl}. \tag{3.1}$$

The necessary and sufficient condition that these be completely integrable, is the vanishing of the curvature tensor B^i_{jkl}. If this is not the case we build sets of equations analogous to (2.5)

$$F^{(\alpha)};\quad \xi^i G^{(\alpha)}_i + \xi^i_{,j} H^{(\alpha)j}_i = 0 \quad (\alpha = 1, 2, \ldots), \tag{3.2}$$

where $G^{(\alpha)}$ and $H^{(\alpha)}$ are tensors depending on B^i_{jkl} and its covariant derivatives of orders not higher than α. If the rank of the matrices of $F^{(1)}, \ldots, F^{(N)}$ and of $F^{(1)}, \ldots, F^{(N+1)}$ is $n^2 + n - r$, the manifold will admit an r-parameter finite continuous group of affine collineations.

Only flat spaces admit a group of affine collineations of the greatest number of parameters, $n^2 + n$. For this case, the coördinates being cartesian, the coefficient of affine connection are zero and (3.1) become

$$\frac{\partial^2\xi^i}{\partial x^j\partial x^k} = 0$$

the solution being

$$\xi^i = a^i_j x + b^i,$$

the finite transformations generated being of the form

$$\bar{x}^i = A^i_j x^j + B^i. \tag{3.3}$$

It can be easily shown that two motions in a Riemann space cannot have the same path-curves.[5] Two collineations may have the same path-curve. In the case of projective collineations the solutions of (2.1) must satisfy additional conditions of the form

$$\xi^i_{,j} = \psi_j \xi^i + \delta^i_j f \tag{3.4}$$

where the vector ψ_j is given by a set of partial differential equations and the scalar f is obtained by contraction.

In case of affine collineations the solutions of (3.1) must satisfy

$$\xi^i_{,j} = \psi_j \xi^i. \tag{3.5}$$

Since in either case the vector ξ^i determines a congruence of paths, we have the theorem:

If a space admits two collineations having the same path-curves, these path-curves must be paths.

4. Another question of importance is whether there exists an affine manifold admitting a given simply transitive group of collineations. This leads to a generalization of Bianchi's theorem for Riemannian geometry and our method is generalized from that used by Professor Eisenhart in establishing Bianchi's theorem.[5]

If $\xi^i_{(\alpha)} (\alpha = 1, \ldots, n)$, $|\xi^i_{(\alpha)}| \neq 0$ are the n vectors of the group, we define the covariant vectors $\eta_i^{(\alpha)}$ by the equations

$$\xi^i_{(\alpha)} \eta_j^{(\alpha)} = \delta^i_j. \tag{4.1}$$

The quantities

$$R^h_{ij} = -\eta_j^{(\alpha)} \frac{\partial \xi^h_{(\alpha)}}{\partial x^i} = \xi^h_{(\alpha)} \frac{\partial \eta_j^{(\alpha)}}{\partial x^i}$$

satisfy the relations

$$R^h_{ij} - R^h_{ji} = -c^\gamma_{\alpha\beta} \xi^h_{(\gamma)} \eta_i^{(\alpha)} \eta_j^{(\beta)}, \tag{4.2}$$

$c^\gamma_{\alpha\beta}$ being the constants of composition of the group, and

$$\frac{\partial R^h_{jk}}{\partial x^l} - \frac{\partial R^h_{jl}}{\partial x^k} = R^h_{pk} R^p_{jl} - R^h_{pl} R^p_{jk}. \tag{4.3}$$

If a manifold is to exist admitting the given group as a group of projective collineations, the equations

$$\frac{\partial \Pi^h_{ij}}{\partial x^k} = \frac{\partial R^h_{ik}}{\partial x^j} - R^h_{im} R^m_{jk} - \Pi^m_{ij} R^h_{mk} + \Pi^h_{im} R^m_{jk} + \Pi^h_{mj} R^m_{ik}$$
$$+ \frac{1}{n+1} \left\{ \delta^h_i \left(R^p_{pm} R^m_{jk} - \frac{\partial R^p_{pk}}{\partial x^j} \right) + \delta^h_j \left(R^p_{pm} R^m_{ik} - \frac{\partial R^p_{pk}}{\partial x^i} \right) \right\} \tag{4.4}$$

must have a solution such that $\Pi^h_{ij} = \Pi^h_{ji}$, $\Pi^h_{hi} = 0$ and which will transform according to the law of transformation of the coefficients of projective connection. By repeated application of (4.2) and (4.3) we find that (4.4) are completely integrable and although in general the solutions will not be coefficients of projective connection, $\infty^{\frac{n(n-1)(n+2)}{2}}$ of them will satisfy all the conditions of the problem.

The same set of vectors $\xi^i_{(\alpha)}$ can serve as the components for a simply transitive group of affine collineations, as the equations

$$\frac{\partial \Gamma^h_{ij}}{\partial x^k} = \frac{\partial R^h_{ik}}{\partial x^j} - R^h_{im}R^m_{jk} - \Gamma^m_{ij}R^h_{mk} + \Gamma^h_{im}R^m_{jk} + \Gamma^h_{mj}R^m_{ik} \tag{4.5}$$

are also completely integrable. The general solutions of (4.5) are coefficients of *asymmetric connection*, $\infty^{\frac{n^2(n+1)}{2}}$ of them being coefficients of affine connection. There exists only one manifold, its coefficients of connection being R^i_{jk}, with respect to which each of the given vectors $\xi^i_{(\alpha)}$ is a parallel vector field.

[1] Eisenhart, L. P., and Knebelman, M. S., these PROCEEDINGS, **13**, 1927, p. 38.

[2] Eisenhart, L. P., *Ibid.*, **8**, 1922, p. 236.

[3] Veblen, O., and Thomas, J. M., *Ann. Math.*, **27**, 1926, p. 287.

[4] Loc. cit., pp. 288–291.

[5] Eisenhart, L. P., *Riemannian Geometry*, **1926**, Chap. VI.

REMARKS ON THE QUANTUM THEORY OF DIFFRACTION

BY P. EHRENFEST AND P. S. EPSTEIN[1]

UNIVERSITY OF LEIDEN (HOLLAND) AND CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *On Fraunhofer Diffraction Phenomena.*—Before attacking the Fresnel problems we shall restate some of the considerations of our last paper, dealing with the Fraunhofer diffraction in a new form suitable for generalization.[2] The main question discussed in that paper was as to the intensity of light diffracted by any optical structure at a given angle to the incident beam. We saw there that the "electronic intensity," ρ of the diffracting system is a function of the space which may be referred to any system of coördinates. Let us use a rectangular cartesian system x, y, z and denote the cosines of the angles between the axes and the direction of the incident ray by α_0, β_0, γ_0.

The general expression for the electronic density can then be given in terms of a three-fold Fourier integral